

# ATIYAH-SUTCLIFFE CONJECTURES FOR ALMOST COLLINEAR CONFIGURATIONS AND SOME NEW CONJECTURES FOR SYMMETRIC FUNCTIONS

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## Abstract

In 2001 Sir M. F. Atiyah formulated a conjecture (*C1*) and later with P. Sutcliffe two stronger conjectures (*C2*) and (*C3*). These conjectures, inspired by physics (spin-statistics theorem of quantum mechanics), are geometrically defined for any configuration of points in the Euclidean three space. The conjecture (*C1*) is proved for  $n = 3, 4$  and for general  $n$  only for some special configurations (M. F. Atiyah, M. Eastwood and P. Norbury, D. Đoković).

In this paper we shall explain some new conjectures for symmetric functions which imply (*C2*) and (*C3*) for almost collinear configurations. Computations up to  $n = 6$  are performed with a help of **Maple** and J. Stembridge's package **SF** for symmetric functions. For  $n = 4$  the conjectures (*C2*) and (*C3*) we have also verified for some infinite families of tetrahedra. This is a joint work with I. Urbiha.

Finally we mention that by minimizing a geometrically defined energy, figuring in these conjectures, one gets a connection to some complicated physical theories, such as Skyrmons and Fullerenes.

## 1 Introduction on Geometric Energies

In this Section we describe some geometric energies, introduced by Atiyah. To construct first geometric energy consider  $n$  distinct ordered points,  $\mathbf{x}_i \in \mathbb{R}^3$  for

$i = 1, \dots, n$ . For each pair  $i \neq j$  define the unit vector

$$\mathbf{v}_{ij} = \frac{\mathbf{x}_j - \mathbf{x}_i}{|\mathbf{x}_j - \mathbf{x}_i|} \quad (1.1)$$

giving the direction of the line joining  $\mathbf{x}_i$  to  $\mathbf{x}_j$ . Now let  $t_{ij} \in \mathbb{CP}^1$  be the point on the Riemann sphere associated with the unit vector  $\mathbf{v}_{ij}$ , via the identification  $\mathbb{CP}^1 \cong S^2$ , realized as stereographic projection. Next, set  $p_i$  to be the polynomial in  $t$  with roots  $t_{ij}$  ( $j \neq i$ ), that is

$$p_i = \alpha_i \prod_{j \neq i} (t - t_{ij}) \quad (1.2)$$

where  $\alpha_i$  is a certain normalization coefficient. In this way we have constructed  $n$  polynomials which all have degree  $n - 1$ , and so we may write

$$p_i = \sum_{j=1}^n d_{ij} t^{j-1}.$$

Finally, let  $d$  be the  $n \times n$  matrix with entries  $d_{ij}$ , and let  $D$  be its determinant

$$D(\mathbf{x}_1, \dots, \mathbf{x}_n) = \det d. \quad (1.3)$$

This geometrical construction is relevant to the Berry-Robbins problem, which is concerned with specifying how a spin basis varies as  $n$  point particles move in space, and supplies a solution provided it can be shown that  $D$  is always non-zero. For  $n = 2, 3, 4$  it can be proved that  $D \neq 0$  (Eastwood and Norbury) and numerical computations suggest that  $|D| \geq 1$  for all  $n$ , with the minimal value  $|D| = 1$  being attained by  $n$  collinear points.

The geometric energy is the  $n$ -point energy defined by

$$E = -\log |D|, \quad (1.4)$$

so minimal energy configurations maximize the modulus of the determinant.

This energy is geometrical in the sense that it only depends on the directions of the lines joining the points, so it is translation, rotation and scale invariant. Remarkably, the minimal energy configurations, studied numerically for all  $n \leq 32$ , are essentially the same as those for the Thomson problem.

## 2 Eastwood–Norbury formulas for Atiyah determinants

In this section we first recall Eastwood–Norbury formula for Atiyah determinant for three or four points in Euclidean three-space. In the case  $n = 3$  the Atiyah determinant reads as

$$\det M_3 = d_3(r_{12}, r_{13}, r_{22}) + 8r_{12}r_{13}r_{22}$$

where

$$d_3(a, b, c) = (a + b - c)(b + c - a)(c + a - b)$$

and  $r_{ij}$  ( $1 \leq i < j \leq 3$ ) is the distance between the  $i^{\text{th}}$  and  $j^{\text{th}}$  point.

In the case  $n = 4$  the Atiyah determinant  $\det M_4$  has real part given by a polynomial (with 248 terms) as follows:

$$\Re(\det M_4) = 64r_{12}r_{13}r_{23}r_{14}r_{24}r_{34} - 4d_3(r_{12}r_{34}, r_{13}r_{24}, r_{14}r_{23}) + A_4 + 288V^2 \quad (2.5)$$

where

$$A_4 = \sum_{l=1}^4 \left( \sum_{(l \neq) i=1}^4 r_{li}((r_{lj} - r_{lk})^2 - r_{jk}^2) \right) d_3(r_{ij}, r_{ik}, r_{jk})$$

(here  $\{j, k\} = \{1, 2, 3, 4\} \setminus \{l, i\}$ ) and  $V$  denotes the volume of the tetrahedron with vertices our four points:

$$\begin{aligned} 144V^2 &= -(r_{12}^4 r_{34}^2 + r_{12}^2 r_{34}^4 + \text{four terms}) + (r_{12}^2 r_{23}^2 r_{34}^2 + \text{eleven terms}) - \\ &= (r_{12}^2 r_{23}^2 r_{13}^2 + \text{three terms}). \end{aligned}$$

We now state two formulas which will be used later:

1. Alternative form of  $A_4$ :

$$A_4 = \sum_{l=1}^4 ((d_3(r_{il}, r_{jl}, r_{kl}) + 8r_{il}r_{jl}r_{kl} + r_{il}(r_{il}^2 - r_{jk}^2) + r_{jl}(r_{jl}^2 - r_{ik}^2) + r_{kl}(r_{kl}^2 - r_{ij}^2)) d_3(r_{ij}, r_{ik}, r_{jk}), \quad (2.6)$$

where for each  $l$  we write  $\{1, 2, 3, 4\} \setminus \{l\} = \{i < j < k\}$ .

2. The sum of the second and the fourth term of (2.5) can be rewritten as

$$\begin{aligned} 288V^2 - 4d_3(r_{12}r_{34}, r_{13}r_{24}, r_{14}r_{23}) &= \\ &= (r_{12} - r_{34})^2(r_{13}^2 r_{24}^2 + r_{14}^2 r_{23}^2 - r_{12}^2 r_{34}^2) + \text{two such terms} + \\ &\quad + 4r_{12}r_{13}r_{23}r_{14}r_{24}r_{34} - \\ &\quad - r_{12}^2 r_{13}^2 r_{23}^2 - r_{12}^2 r_{14}^2 r_{24}^2 - r_{13}^2 r_{14}^2 r_{34}^2 - r_{23}^2 r_{24}^2 r_{34}^2. \end{aligned} \quad (2.7)$$

It is well known that the quantity is always nonpositive.

The imaginary part  $\Im(\det(M_4))$  of Atiyah determinant can be written as a product of  $144V^2$  with a polynomial (with integer coefficients) having 369 terms.

The original Atiyah conjecture in our cases is equivalent to nonvanishing of the determinants  $\det(M_3)$  and  $\det(M_4)$ .

A stronger conjecture of Atiyah and Sutcliffe ([4], Conjecture 2) states in our cases that  $|\det(M_3)| \geq 8r_{12}r_{13}r_{23}$  and  $|\det(M_4)| \geq 64r_{12}r_{13}r_{23}r_{14}r_{24}r_{34}$ .

From the formula (2.5) above, with the help of the simple inequality  $d_3(a, b, c) \leq abc$  (for  $a, b, c \geq 0$ ), Eastwood and Norbury got "almost" the proof of the stronger conjecture by exhibiting the inequality

$$\Re(\det M_4) \geq 60r_{12}r_{13}r_{23}r_{14}r_{24}r_{34}.$$

To remove the word "almost" seems to be not so easy (at present not yet done even for planar configuration of four points).

A third conjecture (stronger than the second) of Atiyah and Sutcliffe ([4], Conjecture 3) can be expressed, in the four point case, in terms of polynomials in the edge lengths as

$$|\det M_4|^2 \geq \prod_{\{i < j < k\} \subset \{1,2,3,4\}} (d_3(r_{ij}, r_{ik}, r_{jk}) + 8r_{ij}r_{ik}r_{jk})$$

where the product runs over the four faces of the tetrahedron, and the expressions for the left hand side are given explicitly.

(cf. `ftp://ftp.maths.adelaide.edu.au/meastwood/maple/points`)

In this paper we study some infinite families of tetrahedra and confirm the strongest conjecture of Atiyah and Sutcliffe for several such infinite families.

## 2.1 Atiyah–Sutcliffe conjecture for (vertically) upright tetrahedra (or pyramids)

By upright tetrahedron we mean a tetrahedron with vertices 1, 2, 3, 4 such that all edges from the vertex 4 have equal lengths. In this case we write

$$r_{23} = a, r_{13} = b, r_{12} = c, r_{14} = r_{24} = r_{34} = d.$$

Now we study the second and the third term of Eastwood–Norbury formula for  $\Re(\det M_4)$ :

$$\begin{aligned} -4d_3(r_{12}r_{34}, r_{13}r_{24}, r_{14}r_{23}) + A_4 &= -4d^3d_3(a, b, c) \\ &+ [c(b^2 + 2bd) + b(c^2 + 2cd) + d((b + c)^2 - a^2)]a^2(2d - a) \\ &+ [c(a^2 + 2ad) + a(c^2 + 2cd) + d((c + a)^2 - b^2)]b^2(2d - b) \\ &+ [b(a^2 + 2ad) + a(b^2 + 2bd) + d((a + b)^2 - c^2)]c^2(2d - c) \\ &+ [d(4d^2 - a^2) + d(4d^2 - b^2) + d(4d^2 - c^2)]d_3(a, b, c) \\ &= dB_1 + B_2 + B_3, \end{aligned}$$

where

$$\begin{aligned} B_1 &= (8d^2 - a^2 - b^2 - c^2)d_3(a, b, c), \\ B_2 &= [a^2b^2c + a^2bc^2](2d - a) + [a^2b^2c + ab^2c^2](2d - b) + [a^2bc^2 + ab^2c^2](2d - c), \\ B_3 &= (4bc + (b + c)^2 - a^2)a^2(2d - a) + (4ac + (a + c)^2 - b^2)b^2(2d - b) + \\ &+ (4ab + (a + b)^2 - c^2)c^2(2d - c) \end{aligned}$$

The term  $B_1$  could still be negative, so we start to employ the geometric constraint on the edge length  $d$ :

$$\begin{aligned} d \geq R &= \text{the circumradius of the base triangle } 123 \\ &= \frac{abc}{\sqrt{(a + b + c)d_3(a, b, c)}} \text{ (by Heron formula)} \end{aligned}$$

Now we split the term  $B_2$  into two parts as follows:

$$B_2 = dB'_2 + B''_2$$

where

$$B'_2 = (a^2b^2c + a^2bc^2) \left(2 - \frac{a}{R}\right) + (a^2b^2c + ab^2c^2) \left(2 - \frac{b}{R}\right) + (a^2bc^2 + ab^2c^2) \left(2 - \frac{c}{R}\right)$$

and

$$B''_2 = (a^2b^2c + a^2bc^2) \left(\frac{ad}{R} - a\right) + (a^2b^2c + ab^2c^2) \left(\frac{bd}{R} - b\right) + (a^2bc^2 + ab^2c^2) \left(\frac{cd}{R} - c\right).$$

**Lemma 2.1** *For any nonnegative real numbers  $a, b, c$  the inequality*

$$\begin{aligned} & [4abc(2abc + (a+b+c)(ab+ac+bc)) - (a^2+b^2+c^2)D_3(a,b,c)]^2 \geq \\ & (a+b+c)^2(a^2b+ab^2+a^2c+ac^2+b^2c+bc^2)^2D_3(a,b,c) \end{aligned} \quad (2.8)$$

where  $D_3(a,b,c) = (a+b+c)(a+b-c)(a-b+c)(-a+b+c) (= (a+b+c)d_3(a,b,c))$ , holds true.

**Proof of Lemma 2.1.**

By assuming  $a \geq b \geq c \geq 0$  and putting  $a = b + h$  and  $b = c + k$ , ( $h \geq 0, k \geq 0$ ), we get (by MAPLE, for example) a polynomial of degree 12 with all coefficients positive (between 1 and 254930). This is a computer verification of the Lemma 2.1. ■

Now, by using  $d \geq R$ , we estimate

$$\begin{aligned} B_1 + B'_2 &= [8d^2 - (a^2 + b^2 + c^2)]d_3(a,b,c) + B'_2 \\ &\geq [8R^2 - (a^2 + b^2 + c^2)]d_3(a,b,c) + B'_2 \\ &= \frac{1}{a+b+c}(8a^2b^2c^2 - (a+b+c)(a^2+b^2+c^2)d_3(a,b,c)) + \\ &\quad + 4(a^2b^2c + a^2bc^2 + ab^2c^2) - \\ &\quad - (a^2b + ab^2 + a^2c + ac^2 + b^2c + bc^2)\sqrt{(a+b+c)d_3(a,b,c)} \end{aligned}$$

But this last quantity is nonnegative. This can be seen by taking square root on both sides of the inequality in the Lemma 2.1. (Note that the quantity inside the brackets in the left hand side of the inequality (2.8) is positive, which can be checked similarly.) So,  $B_1 + B'_2 \geq 0$ . By putting together all what we have proved so far we conclude that the Atiyah–Sutcliffe conjecture

$$\Re(\det M_4) \geq 64r_{12}r_{13}r_{23}r_{14}r_{24}r_{34}$$

is valid for upright tetrahedra.

## 2.2 Atiyah–Sutcliffe conjectures for edge–tangential tetrahedra

By edge–tangential tetrahedron we shall mean any tetrahedron for which there exists a sphere touching all its edges (i.e. its 1–skeleton has an inscribed sphere.) For each  $i$  from 1 to 4 we denote by  $t_i$  the length of the segment (lying on the tangent line) with one endpoint the vertex and the other the point of contact of the tangent line with a sphere. Clearly for the distances  $r_{ij}$  from  $i^{\text{th}}$  to  $j^{\text{th}}$  point we have

$$r_{ij} = t_i + t_j, \quad (1 \leq i < j \leq 4).$$

Now we shall compute all the ingredients appearing in the Eastwood–Norbury formula for  $\Re(\det M_4)$  in terms of elementary symmetric functions of the (tangential) variables  $t_1, t_2, t_3, t_4$  (recall  $e_1 = t_1 + t_2 + t_3 + t_4$ ,  $e_2 = t_1t_2 + t_1t_3 + t_1t_4 + t_2t_3 + t_2t_4 + t_3t_4$ ,  $e_3 = t_1t_2t_3 + t_1t_2t_4 + t_1t_3t_4 + t_2t_3t_4$ ,  $e_4 = t_1t_2t_3t_4$ ).

$$\begin{aligned} 64r_{12}r_{13}r_{23}r_{14}r_{24}r_{34} &= 64 \prod_{1 \leq i < j \leq 4} (t_i + t_j) = 64s_{3,2,1} = \\ &= 64 \begin{vmatrix} e_3 & e_4 & 0 \\ e_1 & e_2 & e_3 \\ 0 & 1 & e_1 \end{vmatrix} = 64e_3e_2e_1 - 64e_4e_1^2 - 64e_3^2 \end{aligned}$$

Here we have used Jacobi–Trudi formula for the triangular Schur function  $s_{3,2,1}$  (see [8], (3.5)). Furthermore we have

$$\begin{aligned} -4d_3(r_{12}r_{34}, r_{13}r_{24}, r_{14}r_{23}) &= 128e_4e_2 - 32e_4e_1^2 - 32e_3^2 \\ 288V^2 &= 128e_4e_2 - 32e_3^2 \end{aligned}$$

In order to compute  $A_4$  we first compute, for fixed  $l$  the following quantities

$$\begin{aligned} d_3(r_{ij}, r_{ik}, r_{jk}) &= 8t_it_jt_k \\ \sum_{(l \neq i)=1}^4 r_{li}((r_{lj} + r_{lk})^2 - r_{jk}^2) &= 4(3t_l(t_1 + t_2 + t_3 + t_4) + (t_it_j + t_it_k + t_jt_k))t_l. \end{aligned}$$

Thus we get:

$$A_4 = 32(3e_1^2 + 4e_2)e_4 = 96e_4e_1^2 + 128e_4e_2.$$

Now we adjust terms in  $\Re(\det M_4)$ , in order to get shorter expression, as follows

$$\begin{aligned} \Re(\det M_4) &= (64r_{12}r_{13}r_{23}r_{14}r_{24}r_{34} - 2 \cdot 288V^2) + \\ &\quad + (-4d_3(r_{12}r_{34}, r_{13}r_{24}, r_{14}r_{23}) - 288V^2) + A_4 + 4 \cdot 288V^2 \\ &= (64e_3e_2e_1 - 64e_4e_1^2 - 256e_4e_2) + (-32e_4e_1^2) + \\ &\quad + (96e_4e_1^2 + 128e_4e_2) + 4 \cdot 288V^2 \\ &= 64e_3e_2e_1 - 128e_4e_2 + 1152V^2 \\ &= 64e_2(e_3e_1 - 2e_4) + 1152V^2 \\ &= 64e_2(2e_4 + m_{211}) + 1152V^2, \end{aligned}$$

where  $m_{211} = t_1^2 t_2 t_3 + \dots$  denotes the monomial symmetric function associated to the partition  $(2, 1, 1)$ .

In order to verify the third conjecture of Atiyah and Sutcliffe

$$|\det M_4|^2 \geq \prod_{\{i < j < k\} \subset \{1, 2, 3, 4\}} (d_3(r_{ij}, r_{ik}, r_{jk}) + 8r_{ij}r_{ik}r_{jk})$$

we note first that

$$\begin{aligned} d_3(r_{ij}, r_{ik}, r_{jk}) + 8r_{ij}r_{ik}r_{jk} &= (8t_i t_j t_k + 8(t_i + t_j)(t_i + t_k)(t_j + t_k)) \\ &= 8(t_i + t_j + t_k)(t_i t_j + t_i t_k + t_j t_k) \end{aligned}$$

and state the following:

**Lemma 2.2** *For any nonnegative real numbers  $t_1, t_2, t_3, t_4 \geq 0$  the following inequality*

$$\begin{aligned} &(t_1 t_2 + t_1 t_3 + t_1 t_4 + t_2 t_3 + t_2 t_4 + t_3 t_4)^2 (2t_1 t_2 t_3 t_4 + m_{211}(t_1, t_2, t_3, t_4))^2 \geq \\ &\geq \prod_{\{i < j < k\} \subset \{1, 2, 3, 4\}} (t_i + t_j + t_k)(t_i t_j + t_i t_k + t_j t_k) \end{aligned} \quad (2.9)$$

holds true.

**Proof of Lemma 2.2.**

The difference between the left hand side and the right hand side of the above inequality (2.9), written in terms of monomial symmetric function is equal to

$$\begin{aligned} LHS - RHS = & m_{6321} + 3m_{6222} + m_{543} + 2m_{5421} + 7m_{5322} + 5m_{5331} + \\ & + 3m_{444} + 7m_{4431} + 8m_{4422} + 8m_{4332} + 3m_{3333} \geq 0 \end{aligned}$$

■

**Remark 2.3** *One may think that the inequality in Lemma 2.2 can be obtained as a product of two simpler inequalities. This is not the case, because the following inequalities hold true:*

$$\begin{aligned} (t_1 t_2 + t_1 t_3 + t_1 t_4 + t_2 t_3 + t_2 t_4 + t_3 t_4)^2 &\leq \prod_{\{i < j < k\} \subset \{1, 2, 3, 4\}} (t_i + t_j + t_k) \\ (2t_1 t_2 t_3 t_4 + m_{211}(t_1, t_2, t_3, t_4))^2 &\geq \prod_{\{i < j < k\} \subset \{1, 2, 3, 4\}} (t_i t_j + t_i t_k + t_j t_k) \end{aligned}$$

Now we continue with verification of the third conjecture of Atiyah and Sutcliffe for edge tangential tetrahedron:

$$\begin{aligned} |\det M_4|^2 &\geq (\Re(\det M_4))^2 \geq [64e_2(2e_4 + m_{211})]^2 \\ &\geq 8^4 \prod_{\{i < j < k\} \subset \{1, 2, 3, 4\}} (t_i + t_j + t_k)(t_i t_j + t_i t_k + t_j t_k) \quad (\text{by Lemma 2.2}) \\ &= \prod_{\{i < j < k\} \subset \{1, 2, 3, 4\}} (d_3(r_{ij}, r_{ik}, r_{jk}) + 8r_{ij}r_{ik}r_{jk}) \end{aligned}$$

so the strongest Atiyah–Sutcliffe conjecture is verified for edge–tangential tetrahedra.

### 2.3 Atiyah–Sutcliffe conjectures for isosceles tetrahedra

By an isosceles tetrahedron we shall mean a tetrahedron in which each pair of opposite edges are equal (hence all triangular faces are congruent). A tetrahedron is isosceles iff the sum of the face angles at each polyhedron vertex is  $180^\circ$ , and iff its insphere and circumsphere are concentric. We now compute the Atiyah determinant for isosceles tetrahedra.

In our notation we have  $r_{23} = r_{14} = a$ ,  $r_{13} = r_{24} = b$ ,  $r_{12} = r_{34} = c$ . By formula (2.6) we get immediately that

$$-4d_3(r_{12}r_{34}, r_{13}r_{24}, r_{14}r_{23}) + 288V^2 = 0.$$

By using our alternative formula (2.6) for the term  $A_4$  in the Eastwood–Norbury formula we get immediately

$$\begin{aligned} A_4 &= \sum_{l=1}^4 (d_3(r_{il}, r_{jl}, r_{kl}) + 8r_{il}r_{jl}r_{kl}) d_3(r_{ij}, r_{ik}, r_{jk}) \\ &= (2d_3(a, b, c) + 8abc)^2 \end{aligned}$$

where for each  $l$  we write  $\{1, 2, 3, 4\} \setminus \{l\} = \{i < j < k\}$ .

The real part of the Atiyah determinant is given by

$$\begin{aligned} \Re(\det(M_4)) &= 64a^2b^2c^2 + 4(d_3(a, b, c) + 8abc)d_3(a, b, c) \\ &= (2d_3(a, b, c) + 8abc). \end{aligned}$$

Now

$$|\det(M_4)|^2 \geq \Re(\det(M_4))^2 = (2d_3(a, b, c) + 8abc)^4 \geq (d_3(a, b, c) + 8abc)^4$$

verifies the third Atiyah–Sutcliffe conjecture for isosceles tetrahedra.

### 2.4 Atiyah determinant for triangles and quadrilaterals via trigonometry

Denote the three points  $x_1, x_2, x_3$  simply by symbols 1, 2, 3 and let  $X, Y$  and  $Z$  denote the angles of the triangle at vertices 1, 2 and 3 respectively. Then we can express the Atiyah determinant  $\det M_3 = d_3(r_{12}, r_{13}, r_{23}) + 8r_{12}r_{13}r_{23}$  as follows

$$\det M_3 = 4r_{12}r_{13}r_{23} \left( \cos^2 \frac{X}{2} + \cos^2 \frac{Y}{2} + \cos^2 \frac{Z}{2} \right).$$

This follows, by using cosine law and sum to product formula for cosine, from the following identity

$$\begin{aligned} d_3(a, b, c) + 8abc &= (a + b - c)(a - b + c)(-a + b + c) + 8abc \\ &= a((b + c)^2 - a^2) + b((c + a)^2 - b^2) + c((a + b)^2 - c^2). \end{aligned}$$



Now we shall translate the Eastwood–Norbury formula for (planar quadrilaterals) into a trigonometric form. Denote the four points  $x_1, x_2, x_3, x_4$  simply by symbols 1, 2, 3, 4 and denote by

$$(X^{(1)}, Y^{(1)}, Z^{(1)}), (X^{(2)}, Y^{(2)}, Z^{(2)}), (X^{(3)}, Y^{(3)}, Z^{(3)}), (X^{(4)}, Y^{(4)}, Z^{(4)})$$

the angles of the triangles 234, 341, 412, 123 in this cyclic order (i.e. the angle of a triangle 412 at vertex 2 is  $Z^{(3)}$  etc.).

Next we denote by  $c_l$ , ( $1 \leq l \leq 4$ ), the sums of cosines squared of half-angles of the  $l$ -th triangle i.e.:

$$c_l := \cos^2 \frac{X^{(l)}}{2} + \cos^2 \frac{Y^{(l)}}{2} + \cos^2 \frac{Z^{(l)}}{2}, \quad l = 1, 2, 3, 4.$$

Similarly, we denote by  $\widehat{c}_l$ , ( $1 \leq l \leq 4$ ), the sum of cosines squared of half-angles at the  $l$ -th vertex of our quadrilateral thus

$$\begin{aligned} \widehat{c}_1 &= \cos^2 \frac{Z^{(2)}}{2} + \cos^2 \frac{Y^{(3)}}{2} + \cos^2 \frac{X^{(4)}}{2} \\ \widehat{c}_2 &= \cos^2 \frac{Z^{(3)}}{2} + \cos^2 \frac{Y^{(4)}}{2} + \cos^2 \frac{X^{(1)}}{2} \\ \widehat{c}_3 &= \cos^2 \frac{Z^{(4)}}{2} + \cos^2 \frac{Y^{(1)}}{2} + \cos^2 \frac{X^{(2)}}{2} \\ \widehat{c}_4 &= \cos^2 \frac{Z^{(1)}}{2} + \cos^2 \frac{Y^{(2)}}{2} + \cos^2 \frac{X^{(3)}}{2} \end{aligned}$$

Then the term  $A_4$  in the Eastwood–Norbury formula can be rewritten as

$$\begin{aligned} A_4 &= \sum_{l=1}^4 (4r_{li}r_{lj}r_{lk}\widehat{c}_l) \cdot 4r_{ij}r_{ik}r_{jk}(c_l - 2) \\ &= 16r_{12}r_{13}r_{23}r_{14}r_{24}r_{34} \sum_{l=1}^4 \widehat{c}_l(c_l - 2). \end{aligned}$$

where for each  $l$  we write  $\{1, 2, 3, 4\} \setminus \{l\} = \{i < j < k\}$ .

In order to rewrite the term  $-4d_3(r_{12}r_{34}, r_{13}r_{24}, r_{14}r_{23})$  into a trigonometric form we recall a theorem of Möbius ([9]) which claims that for any quadrilateral 1234 in a plane the products  $r_{12}r_{34}$ ,  $r_{13}r_{24}$  and  $r_{14}r_{23}$  are proportional to the sides of a triangle whose angles are the differences of angles in the quadrilateral 1234:

$$\begin{aligned} X &= \sphericalangle 243 - \sphericalangle 213 = X^{(4)} - Z^{(1)} \\ Y &= \sphericalangle 341 - \sphericalangle 321 = Y^{(2)} - (-Y^{(4)}) \\ Z &= \sphericalangle 142 - \sphericalangle 132 = -X^{(3)} + Z^{(4)} \end{aligned}$$

Thus

$$-4d_3(r_{12}r_{34}, r_{13}r_{24}, r_{14}r_{23}) = -16r_{12}r_{13}r_{23}r_{14}r_{24}r_{34}(c - 2)$$

where

$$c = \cos^2 \frac{X}{2} + \cos^2 \frac{Y}{2} + \cos^2 \frac{Z}{2}.$$

Thus we have obtained a trigonometric formula for Atiyah determinant of quadrilaterals

$$\begin{aligned} \Re(\det M_4) &= \prod_{1 \leq i < j \leq 4} r_{ij} \left( 64 - 16(c - 2) + 16 \sum_{l=1}^4 \widehat{c}_l (c_l - 2) \right) \\ &= 16 \prod_{1 \leq i < j \leq 4} r_{ij} \left( 6 - c + \sum_{l=1}^4 \widehat{c}_l (c_l - 2) \right) \end{aligned}$$

Now we shall verify Atiyah–Sutcliffe conjecture for cyclic quadrilaterals. In this case, by a well known Ptolemy’s theorem, we see that

$$-4d_3(r_{12}r_{34}, r_{13}r_{24}, r_{14}r_{23}) = 0 \quad (\Leftrightarrow c = 2)$$

By using the equality of angles  $Z^{(2)} = X^{(1)}$ ,  $Z^{(3)} = X^{(2)}$ ,  $Z^{(4)} = X^{(3)}$ ,  $Z^{(1)} = X^{(4)}$  and  $Y^{(1)} + Y^{(3)} = \pi = Y^{(2)} + Y^{(4)}$  (angles with vertex on a circle’s circumference with the same endpoints are equal or suplement of each other) we obtain

$$\begin{aligned} \widehat{c}_1 &= \cos^2 \frac{X^{(1)}}{2} + \cos^2 \frac{Y^{(1)}}{2} + \cos^2 \frac{Z^{(1)}}{2} = c_1 - \cos Y^{(1)}, \\ \widehat{c}_2 &= \cos^2 \frac{X^{(2)}}{2} + \cos^2 \frac{Y^{(2)}}{2} + \cos^2 \frac{Z^{(2)}}{2} = c_2 - \cos Y^{(2)}, \\ \widehat{c}_3 &= \cos^2 \frac{X^{(3)}}{2} + \cos^2 \frac{Y^{(3)}}{2} + \cos^2 \frac{Z^{(3)}}{2} = c_3 - \cos Y^{(3)}, \\ \widehat{c}_4 &= \cos^2 \frac{X^{(4)}}{2} + \cos^2 \frac{Y^{(4)}}{2} + \cos^2 \frac{Z^{(4)}}{2} = c_4 - \cos Y^{(4)}. \end{aligned}$$

Now we have

$$\begin{aligned} \Re(\det M_4) &= \left( \prod_{1 \leq i < j \leq 4} r_{ij} \right) \left( 64 + 16 \sum_{l=1}^4 (c_l - \cos Y^{(l)})(c_l - 2) \right) \\ &\geq \left( \prod_{1 \leq i < j \leq 4} r_{ij} \right) \left( 64 + 16 \sum_{l=1}^4 (c_l - 1)(c_l - 2) \right) \end{aligned}$$

(here we have used that  $2 \leq c_l (\leq \frac{9}{4})$  for each  $l = 1, 2, 3, 4$ )

$$\begin{aligned} &\geq \left( \prod_{1 \leq i < j \leq 4} r_{ij} \right) \left( 64 + 16 \sum_{l=1}^4 (c_l - 2) + 16 \sum_{l=1}^4 (c_l - 2)^2 \right) \\ &\geq \left( \prod_{1 \leq i < j \leq 4} r_{ij} \right) \left( 64 + 16 \sum_{l=1}^4 (c_l - 2) + 4 \left( \sum_{l=1}^4 (c_l - 2) \right)^2 \right) \end{aligned}$$

(by quadratic–arithmetic inequality)

$$\begin{aligned}
&= \left( \prod_{1 \leq i < j \leq 4} r_{ij} \right) \left( \left( 8 + \sum_{l=1}^4 (c_l - 2) \right)^2 + 3 \left( \sum_{l=1}^4 (c_l - 2) \right)^2 \right) \\
&= \left( \prod_{1 \leq i < j \leq 4} r_{ij} \right) \left( \left( \sum_{l=1}^4 c_l \right)^2 + \left( 3 \sum_{l=1}^4 (c_l - 2) \right)^2 \right) \\
&\geq \left( \prod_{1 \leq i < j \leq 4} r_{ij} \right) \left( \sum_{l=1}^4 c_l \right)^2 \geq 16 \sqrt{c_1 c_2 c_3 c_4} \prod_{1 \leq i < j \leq 4} r_{ij}
\end{aligned}$$

by A–G inequality. Finally,

$$\begin{aligned}
|\det M_4|^2 &= |\Re(\det M_4)|^2 \geq 4^4 c_1 c_2 c_3 c_4 \prod_{1 \leq i < j \leq 4} r_{ij}^2 \\
&= \prod_{l=1}^4 (4r_{ij}r_{ik}r_{jk}c_l) = \prod_{l=1}^4 (d_3(r_{ij}, r_{ik}, r_{jk}) + 8r_{ij}r_{ik}r_{jk})
\end{aligned}$$

where for each  $l$  we write  $\{1, 2, 3, 4\} \setminus \{l\} = \{i < j < k\}$ . This finishes verification of Atiyah–Sutcliffe conjectures for cyclic quadrilaterals.

### 3 Almost collinear configurations. Đoković’s approach

#### 3.1 Type (A) configurations

By a type (A) configurations of  $N$  points  $x_1, \dots, x_N$  we shall mean the case when  $N - 1$  of the points  $x_1, \dots, x_N$  are collinear. Set  $n = N - 1$ . In ([6]) Đoković has proved, for configurations of type (A), both the Atiyah conjecture (Theorem 2.1) and the first Atiyah–Sutcliffe conjecture (Theorem 3.1). By using Cartesian coordinates, with  $x_i = (a_i, 0)$ ,  $a_1 < a_2 < \dots < a_n$  and  $x_N = x_{n+1} = (0, b)'$  (with  $b = 1$ ), the normalized Atiyah matrix  $M_{n+1} = M_{n+1}(\lambda_1, \dots, \lambda_n)$  (denoted by  $P$  in [6] when  $b = -1$ ) is given by

$$M_{n+1} = \begin{bmatrix} 1 & \lambda_1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \lambda_2 & \cdots & 0 & 0 \\ 0 & 0 & 1 & & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & & & 1 & \lambda_n \\ (-1)^n e_n & (-1)^{n-1} e_{n-1} & \cdots & \cdots & -e_1 & 1 \end{bmatrix}$$

where  $\lambda_1 = a_1 + \sqrt{a_1^2 + b^2} < \lambda_2 = a_2 + \sqrt{a_2^2 + b^2} < \dots < \lambda_n = a_n + \sqrt{a_n^2 + b^2}$  (with  $b = 1$ ) are positive real numbers and where  $e_k = e_k(\lambda_1, \dots, \lambda_n)$ ,  $1 \leq k \leq n$ ,

is the  $k$ -th elementary symmetric function of  $\lambda_1, \lambda_2, \dots, \lambda_n$ . Its determinant satisfies the inequality

$$\begin{aligned} \det(M_4) &= 1 + \lambda_n e_1 + \lambda_n \lambda_{n-1} e_2 + \dots + \lambda_n \lambda_{n-1} \dots \lambda_1 e_n \\ &\geq 1 + e_1(\lambda_1^2, \dots, \lambda_n^2) + e_2(\lambda_1^2, \dots, \lambda_n^2) + \dots + e_n(\lambda_1^2, \dots, \lambda_n^2) \\ &= \prod_{i=1}^n (1 + \lambda_i^2) \end{aligned}$$

equivalent to the first Atiyah–Sutcliffe conjecture ([4], Conjecture 2). The second Atiyah–Sutcliffe conjecture ([4], Conjecture 3) for configurations of type (A) is equivalent to the following inequality

$$[\det M_{n+1}(\lambda_1, \dots, \lambda_n)]^{n-1} \geq \prod_{k=1}^n \det M_n(\lambda_1, \dots, \lambda_{k-1}, \lambda_{k+1}, \dots, \lambda_n) \quad (3.10)$$

For  $n = 2$  this inequality takes the form

$$1 + \lambda_2 e_1(\lambda_1, \lambda_2) + \lambda_1 \lambda_2 e_2(\lambda_1, \lambda_2) \geq (1 + \lambda_2 e_1(\lambda_2))(1 + \lambda_1 e_1(\lambda_1))$$

i.e.

$$1 + \lambda_2 e_1(\lambda_1, \lambda_2) + \lambda_1 \lambda_2 e_2(\lambda_1, \lambda_2) \geq (1 + \lambda_2^2)(1 + \lambda_1^2). \quad (3.11)$$

This reduces to  $(\lambda_2 - \lambda_1)\lambda_1 \geq 0$ , so it is true.

Even for  $n = 3$  the inequality (3.10) is quite messy thanks to nonsymmetric character of both sides. Knowing that sometimes it is easier to solve a more general problem we followed that path (although we didn't solve the problem in full generality). So let us start with the case  $n = 2$ . If we look at the following inequality

$$1 + X_1(\xi_1 + \xi_2) + X_1 X_2 \xi_1 \xi_2 \geq (1 + X_1 \xi_1)(2 + X_2 \xi_2)$$

which is clearly true if  $X_1 \geq X_2 \geq 0$  and  $\xi_1, \xi_2 \geq 0$  we obtain the inequality (3.11) simply by a specialization  $X_1 = \xi_1 = \lambda_2$ ,  $X_2 = \xi_2 = \lambda_1$ . So we proceed as follows:

Let  $\xi_1, \dots, \xi_n, X_1, \dots, X_n, n \geq 1$  be two sets of commuting indeterminates. For any  $l, 1 \leq l \leq n$  and any sequences  $1 \leq i_1 \leq \dots \leq i_l \leq n, 1 \leq j_1 \leq \dots \leq j_l \leq n$  we define polynomials  $\Psi_J^I = \Psi_{j_1, \dots, j_l}^{i_1, \dots, i_l} \in \mathbb{Q}[\xi_1, \dots, \xi_n, X_1, \dots, X_n]$  as follows:

$$\Psi_J^I := \sum_{k=0}^l e_k(\xi_{j_1}, \xi_{j_2}, \dots, \xi_{j_l}) X_{i_1} X_{i_2} \dots X_{i_k}, \quad (l \geq 1), \quad \Psi_\emptyset^\emptyset := 1 \quad (j = 0)$$

where  $e_k$  is the  $k$ -th elementary symmetric function.

In particular we have

$$\begin{aligned} \Psi_j^i &= 1 + \xi_j X_i, \\ \Psi_{j_1 j_2}^{i_1 i_2} &= 1 + (\xi_{j_1} + \xi_{j_2}) X_{i_1} + \xi_{j_1} \xi_{j_2} X_{i_1} X_{i_2}, \\ \Psi_{j_1 j_2 j_3}^{i_1 i_2 i_3} &= 1 + (\xi_{j_1} + \xi_{j_2} + \xi_{j_3}) X_{i_1} + (\xi_{j_1} \xi_{j_2} + \xi_{j_1} \xi_{j_3} + \xi_{j_2} \xi_{j_3}) X_{i_1} X_{i_2} + \\ &\quad + \xi_{j_1} \xi_{j_2} \xi_{j_3} X_{i_1} X_{i_2} X_{i_3}, \\ &\text{etc.} \end{aligned}$$

The polynomials  $\Psi_J^I$  are symmetric w.r.t.  $\xi_{j_1}, \xi_{j_2}, \dots, \xi_{j_l}$ , but nonsymmetric w.r.t.  $X_{i_1}, X_{i_2}, \dots, X_{i_l}$ . These polynomials when restricted to nonnegative arguments such that  $X_{i_1} \geq X_{i_2} \geq \dots \geq X_{i_l} \geq 0$ ,  $\xi_{j_1}, \xi_{j_2}, \dots, \xi_{j_l} \geq 0$  obey some intriguing inequalities which are not yet proved in full generality. In turn they generalize some special cases of not yet proven conjectures of Atiyah and Sutcliffe on configurations of points in three dimensional Euclidean space (the former inequalities are not yet proven even for the case of  $n = 4$  points).

Let us now formulate a conjecture which implies the strongest Atiyah–Sutcliffe’s conjecture for almost collinear configurations of points (all but one point are collinear, called type(A) in [6]).

Our conjecture reads as follows:

**Conjecture 3.1** *For any  $n \geq 1$ , let  $X_{i_1} \geq X_{i_2} \geq \dots \geq X_{i_l} \geq 0$ ,  $\xi_{j_1}, \xi_{j_2}, \dots, \xi_{j_l} \geq 0$ , be any nonnegative real numbers. Then*

$$(\Psi_{12 \dots n}^{12 \dots n})^{n-1} \geq \prod_{k=1}^n \Psi_{12 \dots \hat{k} \dots n}^{12 \dots \hat{k} \dots n}$$

where  $12 \dots \hat{k} \dots n$  denotes the sequence  $12 \dots (k-1)(k+1) \dots n$ . The equality obviously holds true iff  $X_1 = X_2 = \dots = X_n$ .

To illustrate the Conjecture (3.1) we consider first the cases  $n = 2$  and  $n = 3$ .

**Case  $n = 2$ :** We have

$$\begin{aligned} \Psi_{12}^{12} &= 1 + (\xi_1 + \xi_2)X_1 + \xi_1\xi_2X_1X_2 = \\ &= 1 + \xi_1X_1 + \xi_2X_2 + \xi_1\xi_2X_1X_2 + (X_1 - X_2)\xi_2 = \\ &= (1 + \xi_1X_1)(1 + \xi_2X_2) + \xi_2(X_1 - X_2) \geq \\ &\geq (1 + \xi_1X_1)(1 + \xi_2X_2) = \Psi_1^1\Psi_2^2. \end{aligned}$$

**Case  $n = 3$ :** We first write  $\Psi_{123}^{123}$  in two different ways:

$$\Psi_{123}^{123} = \xi_2(X_1 - X_2) + \widehat{\Psi}_{123}^{123} \quad \text{and} \quad \Psi_{123}^{123} = \xi_3(X_1 - X_2) + \widehat{\Psi}_{123}^{123}.$$

Note that  $\widehat{\Psi}_{123}^{123}$  is obtained from  $\Psi_{123}^{123}$  by replacing the linear term  $\xi_2X_1$  by  $\xi_2X_2$ , hence all its coefficients are nonnegative.

The left hand side of the Conjecture (3.1)  $L_3$  can be rewritten as follows:

$$\begin{aligned} L_3 &= (\Psi_{123}^{123})^2 = (\xi_2(X_1 - X_2) + \widehat{\Psi}_{123}^{123})\Psi_{123}^{123} \\ &= \xi_2(X_1 - X_2)\Psi_{123}^{123} + \widehat{\Psi}_{123}^{123}\Psi_{123}^{123} \\ &= \xi_2(X_1 - X_2)\Psi_{123}^{123} + \widehat{\Psi}_{123}^{123}(\xi_3(X_1 - X_2) + \widehat{\Psi}_{123}^{123}) \\ &= L'_3(X_1 - X_2) + \widehat{\Psi}_{123}^{123}\widehat{\Psi}_{123}^{123} \end{aligned}$$

where  $L'_3 = \xi_2\Psi_{123}^{123} + \xi_3\widehat{\Psi}_{123}^{123}$  is a positive polynomial.

Now we have

$$L_3 \geq \widehat{L}_3 := \widehat{\Psi}_{123}^{123} \widehat{\Psi}_{123}^{123}.$$

By using the formula

$$\widehat{\Psi}_{123}^{123} = \Psi_{13}^{12} + \xi_2 X_2 \Psi_{13}^{13} = (\Psi_2^2 - 1) \Psi_{13}^{13} + \Psi_{13}^{12}$$

we can rewrite  $\widehat{L}_3$  as

$$\begin{aligned} \widehat{L}_3 &= [(\Psi_{13}^{12} - \Psi_{13}^{13}) + \Psi_2^2 \Psi_{13}^{13}] \widehat{\Psi}_{123}^{123} \\ &= \xi_1 \xi_3 X_1 (X_2 - X_3) \widehat{\Psi}_{123}^{123} + \Psi_{13}^{13} (\Psi_2^2 \widehat{\Psi}_{123}^{123}) \end{aligned}$$

The last term in parenthesis can be written as

$$\begin{aligned} \Psi_2^2 \widehat{\Psi}_{123}^{123} &= \Psi_{12}^{12} \Psi_{23}^{23} + \Psi_2^1 (\Psi_{23}^{22} - \Psi_{23}^{23}) \\ &= \Psi_{12}^{12} \Psi_{23}^{23} + \xi_2 \xi_3 X_2 (X_2 - X_3) \Psi_2^1, \end{aligned}$$

so we get

$$\widehat{L}_3 = L_3'' (X_2 - X_3) + \Psi_{12}^{12} \Psi_{13}^{13} \Psi_{23}^{23}$$

where  $L_3''$  denotes the positive polynomial

$$L_3'' = \xi_1 \xi_3 X_1 \widehat{\Psi}_{123}^{123} + \xi_2 \xi_3 X_2 \Psi_2^1 \Psi_{13}^{13}.$$

We now have an explicit formula for  $L_3$ :

$$L_3 = L_3' (X_1 - X_2) + L_3'' (X_2 - X_3) + \Psi_{12}^{12} \Psi_{13}^{13} \Psi_{23}^{23}$$

with  $L_3', L_3''$  positive polynomials, which together with  $X_1 \geq X_2 \geq X_3 (\geq 0)$  implies that

$$L_3 \geq R_3 := \Psi_{12}^{12} \Psi_{13}^{13} \Psi_{23}^{23}$$

and the Conjecture (3.1) ( $n = 3$ ) is proved.

In fact we have proven an instance  $n = 3$   $\widehat{L}_3 \geq R_3$  of a stronger conjecture which we are going to formulate now. Let  $2 \leq k \leq n$ . We define the modified polynomials  $\widehat{\Psi}_{12 \dots \underline{k} \dots n}^{12 \dots k \dots n}$  as follows:

$$\widehat{\Psi}_{12 \dots \underline{k} \dots n}^{12 \dots k \dots n} := \xi_k (X_2 - X_1) + \Psi_{12 \dots n}^{12 \dots n}$$

obtained from  $\Psi_{12 \dots n}^{12 \dots n}$  by replacing only one term  $\xi_k X_1$  by  $\xi_k X_2$ , hence  $\widehat{\Psi}_{12 \dots \underline{k} \dots n}^{12 \dots k \dots n}$  are still positive. Let us introduce the following notation:

$$\widehat{L}_n := \prod_{k=2}^n \widehat{\Psi}_{12 \dots \underline{k} \dots n}^{12 \dots k \dots n} ; \quad R_n := \prod_{k=1}^n \Psi_{12 \dots \underline{k} \dots n}^{12 \dots k \dots n}.$$

Then clearly  $L_n := (\Psi_{12 \dots n}^{12 \dots n})^{n-1} \geq \widehat{L}_n$ . Now our stronger conjecture reads as

**Conjecture 3.2**

$$\widehat{L}_n \geq R_n \quad (n \geq 1)$$

with equality iff  $X_2 = X_3 = \dots = X_n$ .

More generally, we conjecture that the difference  $\widehat{L}_n - R_n$  is a polynomial in the differences  $X_2 - X_3, X_3 - X_4, \dots, X_{n-1} - X_n$  with coefficients in  $\mathbb{Z}_{\geq 0}[X_1, \dots, X_n, \xi_1, \dots, \xi_n]$ .

**Proposition 3.3**

$$L_n = L'_n(X_1 - X_2) + \widehat{L}_n$$

for some positive polynomial  $L'_n$ .

**Proof of Proposition 3.3.**

$$\begin{aligned} L_n &= (\Psi_{12\dots n}^{12\dots n})^{n-1} = (\xi_2(X_1 - X_2) + \widehat{\Psi}_{12\dots n}^{12\dots n})(\Psi_{12\dots n}^{12\dots n})^{n-2} \\ &= \xi_2(X_1 - X_2)(\Psi_{12\dots n}^{12\dots n})^{n-2} + \widehat{\Psi}_{12\dots n}^{12\dots n}(\xi_3(X_1 - X_2) + \widehat{\Psi}_{123\dots n}^{123\dots n})(\Psi_{12\dots n}^{12\dots n})^{n-3} \\ &= \xi_2(X_1 - X_2)(\Psi_{12\dots n}^{12\dots n})^{n-2} + \xi_3(X_1 - X_2)\widehat{\Psi}_{12\dots n}^{12\dots n}(\Psi_{12\dots n}^{12\dots n})^{n-3} + \\ &\quad + \widehat{\Psi}_{12\dots n}^{12\dots n}\widehat{\Psi}_{123\dots n}^{123\dots n}(\Psi_{12\dots n}^{12\dots n})^{n-3} \\ &\vdots \\ &= (\sum_{k=1}^{n-1} \xi_{k+1}(\prod_{j=2}^k \widehat{\Psi}_{12\dots j\dots n}^{12\dots j\dots n})(\Psi_{12\dots n}^{12\dots n})^{n-k})(X_1 - X_2) + \prod_{j=2}^n \widehat{\Psi}_{12\dots j\dots n}^{12\dots j\dots n}. \end{aligned}$$

■

Now we turn to study the quotient

$$\frac{L_n}{R_n} = \frac{(\Psi_{1\dots n}^{1\dots n})^{n-1}}{\prod_{k=1}^n \Psi_{1\dots \widehat{k} \dots n}^{1\dots \widehat{k} \dots n}}$$

by studying the growth behaviour of quotients of its factors  $\Psi_{1\dots n}^{1\dots n} / \Psi_{1\dots \widehat{k} \dots n}^{1\dots \widehat{k} \dots n}$  w.r.t. any of its arguments  $X_r$ ,  $1 \leq r \leq n$ .

In the following theorem we obtain an explicit formula for the numerators of the logarithmic derivatives w.r.t.  $X_r$ , ( $1 \leq r \leq n, r \neq k$ ) of the quantities  $\Psi_{1\dots n}^{1\dots n}/\Psi_{1\dots \hat{k}\dots n}^{1\dots \hat{k}\dots n}$ . From this formulas we get some monotonicity properties which enable us to state some new (refined) conjectures later on.

**Theorem 3.4** *Let*

$$\Delta_r := \partial_{X_r} \Psi_{1\dots n}^{1\dots n} \cdot \Psi_{1\dots \hat{k}\dots n}^{1\dots \hat{k}\dots n} - \Psi_{1\dots n}^{1\dots n} \cdot \partial_{X_r} \Psi_{1\dots \hat{k}\dots n}^{1\dots \hat{k}\dots n}, \quad (1 \leq r \leq n). \quad (3.12)$$

*Then we have the following explicit formulas*

(i) *for any  $r$ ,  $1 \leq r < k(\leq n)$  we have*

$$\begin{aligned} \Delta_r = & \xi_k \sum_{0 \leq i < r \leq j \leq n} s_{(2^i 1^{j-i-1})}^{(k)} X_1^2 \cdots X_i^2 X_{i+1} \cdots \hat{X}_k \cdots X_j + \\ & + \sum_{0 \leq i < r, k \leq j < n} e_i e_j^{(k)} X_1^2 \cdots X_i^2 X_{i+1} \cdots \hat{X}_r \cdots \hat{X}_k \cdots X_j (X_k - X_{j+1}) \end{aligned}$$

(ii) *for any  $r$ ,  $(1 \leq) k < r \leq n$  we have*

$$\begin{aligned} \Delta_r = & - \left( \sum_{0 \leq i < r \leq j \leq n} s_{(2^i 1^{j-i-1})}^{(k)} X_1^2 \cdots X_i^2 X_{i+1} \cdots \hat{X}_k \cdots \hat{X}_r \cdots X_j + \right. \\ & \left. + \sum_{0 \leq i < k, r \leq j < n} e_i^{(k)} e_j X_1^2 \cdots X_i^2 X_{i+1} \cdots \hat{X}_k \cdots \hat{X}_r \cdots X_j (X_{j+1} - X_k) \right) \end{aligned}$$

where  $s_{\lambda}^{(k)}$  denotes the  $\lambda$ -th Schur function of  $\xi_1, \dots, \xi_{k-1}, \xi_{k+1}, \dots, \xi_n$  ( $\xi_k$  omitted).

**Proof of Theorem 3.4.**

(i) For any  $r$ ,  $1 \leq r < k(\leq n)$  we find explicitly a formula for as follows. We shall use notations  $X_{1..i} := X_1 X_2 \cdots X_i$ , for multilinear monomials and  $e_i := e_i(\xi_1, \dots, \xi_n)$ ,  $e_i^{(k)} = e_i(\xi_1, \dots, \hat{\xi}_k, \dots, \xi_n)$  for the elementary symmetric functions (here  $k$  is fixed). Then we can rewrite our basic quantities

$$\Psi_{1\dots n}^{1\dots n} := \sum_{i=0}^n e_i X_{1..i} \quad (3.13)$$

$$\begin{aligned} \Psi_{1\dots \hat{k}\dots n}^{1\dots \hat{k}\dots n} &:= \sum_{i=0}^{k-1} e_i^{(k)} X_{1..i} + \frac{1}{X_k} \sum_{i=0}^{n-1} e_i^{(k)} X_{1..i+1} = \\ &= \sum_{i=0}^{n-1} e_i^{(k)} X_{1..i} + \frac{1}{X_k} \sum_{i=0}^{n-1} e_i^{(k)} X_{1..i} (X_{i+1} - X_k) \end{aligned} \quad (3.14)$$

For the derivatives we get immediately

$$\partial_{X_r} \Psi_{1\dots n}^{1\dots n} = \frac{1}{X_r} \sum_{i=r}^n e_i X_{1..i} = \frac{1}{X_r} \left( \Psi_{1\dots n}^{1\dots n} - \sum_{i=0}^{r-1} e_i X_{1..i} \right) \quad (3.15)$$



$$\partial_{X_r} \Psi_{1 \dots \widehat{k} \dots n}^{1 \dots \widehat{k} \dots n} = \frac{1}{X_r} \sum_{i=r}^{n-1} e_i^{(k)} X_{1..i} + \frac{1}{X_k X_r} \sum_{i=k}^{n-1} e_i^{(k)} X_{1..i} (X_{i+1} - X_k) \quad (3.16)$$

$$= \frac{1}{X_r} \left( \Psi_{1 \dots \widehat{k} \dots n}^{1 \dots \widehat{k} \dots n} - \sum_{i=0}^{r-1} e_i^{(k)} X_{1..i} \right) \quad (3.17)$$

By plugging (3.15) and (3.17) into (3.12) we obtain

$$X_r \Delta_r = \Psi_{1 \dots n}^{1 \dots n} \left( \sum_{i=0}^{r-1} e_i^{(k)} X_{1..i} \right) - \Psi_{1 \dots \widehat{k} \dots n}^{1 \dots \widehat{k} \dots n} \left( \sum_{i=0}^{r-1} e_i X_{1..i} \right) =$$

and after simple cancelation, by invoking (3.14) we get

$$= \left( \sum_{j=r}^n e_j X_{1..j} \right) \left( \sum_{i=0}^{r-1} e_i^{(k)} X_{1..i} \right) - \left( \sum_{j=r}^{n-1} e_j^{(k)} X_{1..j} + \frac{1}{X_k} \sum_{j=k}^{n-1} e_j^{(k)} X_{1..j} (X_{j+1} - X_k) \right) \left( \sum_{i=0}^{r-1} e_i X_{1..i} \right)$$

i.e.

$$X_r \Delta_r = \sum_{0 \leq i < r \leq j \leq n} (e_j e_i^{(k)} - e_i e_j^{(k)}) X_{1..i} X_{1..j} + \frac{1}{X_k} \sum_{0 \leq i < r, k \leq j < n} e_i e_j^{(k)} X_{1..i} X_{1..j} (X_k - X_{j+1})$$

If we use a simple identity  $e_j = e_j^{(k)} + \xi_k e_{j-1}^{(k)}$ , we can identify the quantity

$$\begin{aligned} e_j e_i^{(k)} - e_i e_j^{(k)} &= (e_j^{(k)} + \xi_k e_{j-1}^{(k)}) e_i^{(k)} - (e_i^{(k)} + \xi_k e_{i-1}^{(k)}) e_j^{(k)} = \\ &= \begin{vmatrix} e_{j-1}^{(k)} & e_j^{(k)} \\ e_{i-1}^{(k)} & e_i^{(k)} \end{vmatrix} \xi_k = s_{2^i 1^{j-i-1}} \xi_k \end{aligned}$$

Thus in this case ( $1 \leq r < k$ ) we obtain a formula

$$\begin{aligned} \Delta_r &= \xi_k \sum_{0 \leq i < r \leq j \leq n} s_{(j-1, i)(k)}^{(k)} X_1^2 \cdots X_i^2 X_{i+1} \cdots \widehat{X}_k \cdots X_j + \\ &\quad + \sum_{0 \leq i < r, k \leq j < n} e_i e_j^{(k)} X_1^2 \cdots X_i^2 X_{i+1} \cdots \widehat{X}_r \cdots \widehat{X}_k \cdots X_j (X_k - X_{j+1}) \end{aligned}$$

(where  $e_j^{(k)} = e_j = e_j(\xi_1, \dots, \widehat{\xi}_k, \dots, \xi_n)$ ) in terms of a Schur function (of arguments  $\xi_1, \dots, \widehat{\xi}_k, \dots, \xi_n$ ) corresponding to a transpose  $(2^i 1^{j-i-1})$  of a partition  $(j-i, i)$  (cf. Jacobi-Trudi formula, I 3.5 in [8]).

(ii) For any  $r$ ,  $(1 \leq) k < r \leq n$ . In this case we use

$$\partial_{X_r} \Psi_{1 \dots \widehat{k} \dots n}^{1 \dots \widehat{k} \dots n} = \frac{1}{X_k X_r} \sum_{j=r-1}^{n-1} e_j^{(k)} X_{1..j+1}$$

$$\begin{aligned}
\Psi_{1\ldots\widehat{k}\ldots n}^{1\ldots\widehat{k}\ldots n} &= \sum_{i=0}^{k-1} e_i^{(k)} X_{1..i} + \frac{1}{X_k} \sum_{i=k}^{n-1} e_i^{(k)} X_{1..i+1} = \\
&= \frac{1}{X_k} \left( \sum_{i=0}^{k-1} X_{1..i} (X_k - X_{i+1}) + \sum_{i=0}^{n-1} e_i^{(k)} X_{1..i} \right)
\end{aligned}$$

By plugging this into (3.12) we get

$$\begin{aligned}
X_k X_r \Delta_r &= \left( \sum_{j=r}^n e_j X_{1..j} \right) \left( \sum_{i=0}^{k-1} e_i^{(k)} X_{1..i} (X_k - X_{i+1}) + \sum_{i=0}^{n-1} e_i^{(k)} X_{1..i+1} \right) - \\
&\quad - \left( \sum_{j=0}^{r-1} e_j X_{1..j} + \sum_{j=r}^n e_j X_{1..j} \right) \left( \sum_{i=r-1}^{n-1} e_i^{(k)} X_{1..i+1} \right) \\
&= \left( \sum_{i=0}^{r-2} e_i^{(k)} X_{1..i+1} \right) \left( \sum_{j=r}^n e_j X_{1..j} \right) - \left( \sum_{i=0}^{r-1} e_i X_{1..i} \right) \left( \sum_{j=r-1}^{n-1} e_j^{(k)} X_{1..j+1} \right) + \\
&\quad + \sum_{i=0}^{k-1} \sum_{j=r}^n e_i^{(k)} e_j X_{1..i} X_{1..j} (X_k - X_{i+1}) \\
&= \left( \sum_{i=1}^{r-1} e_{i-1}^{(k)} X_{1..i} \right) \left( \sum_{j=r}^n e_j X_{1..j} \right) - \left( \sum_{i=0}^{r-1} e_i X_{1..i} \right) \left( \sum_{j=r}^n e_{j-1}^{(k)} X_{1..j} \right) + \\
&\quad + \sum_{i=0}^{k-1} \sum_{j=r}^n e_i^{(k)} e_j X_{1..i} X_{1..j} (X_k - X_{i+1})
\end{aligned}$$

By using a formula for elementary symmetric functions ( $e_i = e_i^{(k)} + \xi_k e_{i-1}^{(k)}$ ) we can write in terms of Schur functions (of arguments  $\xi_1, \dots, \xi_{k-1}, \xi_{k+1}, \dots, \xi_n$ ), where  $\lambda'$  is a conjugate of  $\lambda$ .

$$e_{i-1}^{(k)} e_j - e_i e_{j-1}^{(k)} = e_{i-1}^{(k)} e_j^{(k)} - e_i^{(k)} e_{j-1}^{(k)} = - \begin{vmatrix} e_{j-1}^{(k)} & e_j^{(k)} \\ e_{i-1}^{(k)} & e_i^{(k)} \end{vmatrix} = -s_{2^i 1^{j-i-1}}^{(k)} = -s_{(j-1, i)}^{(k)},$$

Thus we obtain a formula

$$\begin{aligned}
\Delta_r &= - \left( \sum_{0 \leq i < r \leq j \leq n} s_{(j-1, i)}^{(k)} X_1^2 \cdots X_i^2 X_{i+1} \cdots \widehat{X}_k \cdots \widehat{X}_r \cdots X_j + \right. \\
&\quad \left. + \sum_{0 \leq i < k, r \leq j < n} e_i^{(k)} e_j X_1^2 \cdots X_i^2 X_{i+1} \cdots \widehat{X}_k \cdots \widehat{X}_r \cdots X_j (X_{j+1} - X_k) \right)
\end{aligned}$$

**Corollary 3.5** *Let  $X_1 \geq \dots \geq X_n \geq 0$ ,  $\xi_1, \dots, \xi_n \geq 0$  be as before. Then*

(i) *for any  $r$ ,  $1 \leq r < k$  ( $\leq n$ ) we have*

$$\frac{\Psi_{1\ldots n}^{1\ldots n}}{\Psi_{1\ldots\widehat{k}\ldots n}^{1\ldots\widehat{k}\ldots n}} \geq \frac{\Psi_{1\ldots r}^{1\ldots r+1} \Psi_{r+1\ldots n}^{r+1\ldots n}}{\Psi_{1\ldots r}^{1\ldots r+1} \Psi_{r+1\ldots\widehat{k}\ldots n}^{r+1\ldots\widehat{k}\ldots n}}$$

(ii) for any  $r$ ,  $(1 \leq) k < r (\leq n)$  we have

$$\frac{\Psi_{1\dots n}^{1\dots n}}{\Psi_{1\dots \hat{k}\dots n}^{1\dots \hat{k}\dots n}} \geq \frac{\Psi_{1\dots r-1 \dots r-1 \dots r \dots n}^{1\dots r-1 \dots r-1 \dots r \dots n}}{\Psi_{1\dots \hat{k}\dots r-1 \dots r-1 \dots r \dots n}^{1\dots \hat{k}\dots r-1 \dots r-1 \dots r \dots n}}$$

Now we illustrate how to use Corollary 3.5 to prove our Conjecture 3.1 for  $n = 2, 3, 4$  and 5.

**Case  $n = 2$**

$$Q_2 := \frac{\Psi_{12}^{12}}{\Psi_1^1 \Psi_2^2} \geq \frac{\Psi_{12}^{22}}{\Psi_1^2 \Psi_2^2} = 1 \text{ (by (i))}$$

**Case  $n = 3$**

$$\begin{aligned} Q_3 &:= \frac{\Psi_{123}^{123} \Psi_{123}^{123}}{\Psi_{12}^{12} \Psi_{13}^{13} \Psi_{23}^{23}} \geq \frac{\Psi_{123}^{223} \Psi_{123}^{123}}{\Psi_{12}^{22} \Psi_{13}^{13} \Psi_{23}^{23}} \geq \frac{\Psi_{123}^{223} \Psi_{123}^{223}}{\Psi_{12}^{22} \Psi_{13}^{13} \Psi_{23}^{23}} \text{ (by (i))} \\ &\geq \frac{\Psi_{123}^{222} \Psi_{123}^{223}}{\Psi_{12}^{22} \Psi_{13}^{22} \Psi_{23}^{23}} \geq \frac{\Psi_{123}^{222} \Psi_{123}^{222}}{\Psi_{12}^{22} \Psi_{13}^{22} \Psi_{23}^{23}} = 1 \text{ (by (ii))} \end{aligned}$$

**Case  $n = 4$**

$$Q_4 := \frac{(\Psi_{1234}^{1234})^3}{\Psi_{123}^{123} \Psi_{124}^{124} \Psi_{134}^{134} \Psi_{234}^{234}} \geq \dots \geq \frac{\Psi_{1234}^{2244} (\Psi_{1234}^{2224})^2}{\Psi_{123}^{224} \Psi_{124}^{224} \Psi_{134}^{224} \Psi_{234}^{224}} (\geq 1)$$

This last inequality follows from the following symmetric function identity:

$$\begin{aligned} &\Psi_{1234}^{2244} (\Psi_{1234}^{2224})^2 - \Psi_{123}^{224} \Psi_{124}^{224} \Psi_{134}^{224} \Psi_{234}^{224} = \\ &X_2^2 X_4^4 m_{2222} + 2X_2^2 X_4^3 m_{2221} + X_2^2 X_4^2 m_{222} + 3X_2^2 X_4^2 m_{2211} + X_2^2 X_4 m_{221} \\ &+ 4X_2^2 X_4 m_{2111} + X_2^2 m_{211} + X_2 (3X_2 + 2X_4) m_{1111} + X_2 m_{111} \end{aligned}$$

where  $m_\lambda = m_\lambda(\xi_1, \xi_2, \xi_3, \xi_4)$  are the monomial symmetric functions.

**Case  $n = 5$**

$$Q_5 := \frac{(\Psi_{1\dots 5}^{1\dots 5})^4}{\prod_{k=1}^5 \Psi_{1\dots \hat{k}\dots 5}^{1\dots \hat{k}\dots 5}} \geq \dots \geq \frac{(\Psi_{12345}^{22244} \Psi_{12345}^{22444})^2}{\Psi_{1234}^{2244} \Psi_{1235}^{2244} \Psi_{1245}^{2244} \Psi_{1345}^{2244} \Psi_{2345}^{2244}} (\geq 1)$$

The last inequality is equivalent to an explicit symmetric function identity with all coefficients (w.r.t. monomial basis) positive.

Now we state our stronger conjecture.

**Conjecture 3.6** (for symmetric functions)

(a) For  $n$  even we have

$$\Psi_{1 \ 2 \ \dots \ n-1 \ n}^2 \ 2 \ 4 \ 4 \dots n \ n \left( \prod_{k=1}^{n/2} \Psi_{1 \ 2 \ 3 \ 4 \ \dots \ 2k \ 2k \ 2k \dots n-2 \ n-2 \ n}^2 \right)^2 \geq \prod_{k=1}^n \Psi_{1 \ 2 \ \dots \ \hat{k} \ \dots \ n-1 \ n}^2 \ 2 \ 4 \ 4 \dots n-2 \ n-2 \ n$$

(b) For  $n$  odd we have

$$\left( \prod_{k=1}^{\lfloor n/2 \rfloor} \Psi_{1 \ 2 \ 3 \ 4 \dots 2k \ 2k \ 2k \dots n-1 \ n-1}^{2 \ 2 \ 4 \ 4 \dots 2k \ 2k \ 2k \dots n-1 \ n-1} \right)^2 \geq \prod_{k=1}^n \Psi_{1 \ 2 \dots \hat{k} \dots n}^{2 \ 2 \ 4 \ 4 \dots n-1 \ n-1}$$

Now we motivate another inequalities for symmetric functions which also refine the strongest Atiyah–Sutcliffe conjecture for configurations of type (A). Let  $n = 3$ . We apply Corollary 3.5 by using steps (ii) only.

$$Q_3 := \frac{\Psi_{12}^{123} \Psi_{13}^{123} \Psi_{23}^{123}}{\Psi_{12}^{12} \Psi_{13}^{13} \Psi_{23}^{23}} \geq \frac{\Psi_{12}^{113} \Psi_{13}^{123} \Psi_{23}^{123}}{\Psi_{12}^{12} \Psi_{13}^{13} \Psi_{23}^{13}} \geq \frac{\Psi_{12}^{112} \Psi_{13}^{123} \Psi_{23}^{123}}{\Psi_{12}^{12} \Psi_{13}^{12} \Psi_{23}^{13}} \geq \frac{\Psi_{12}^{112} \Psi_{13}^{122} \Psi_{23}^{122}}{\Psi_{12}^{12} \Psi_{13}^{12} \Psi_{23}^{12}} \geq 1$$

The last inequality is equivalent to nonnegativity of the expression

$$\Psi_{123}^{112} \Psi_{123}^{122} - \Psi_{12}^{12} \Psi_{13}^{12} \Psi_{23}^{12} \quad (= X_1(X_1 - X_2)^2 \xi_1 \xi_2 \xi_3 \geq 0).$$

Similarly, for  $n = 4$ , the symmetric function inequality stronger than  $Q_4 \geq 1$  would be the following

$$\Psi_{1234}^{1123} \Psi_{1234}^{1223} \Psi_{1234}^{1233} \geq \Psi_{123}^{123} \Psi_{124}^{123} \Psi_{134}^{123} \Psi_{234}^{123}$$

Now we state a general conjecture for symmetric functions which imply the strongest Atiyah–Sutcliffe conjecture for almost collinear type (A) configurations.

**Conjecture 3.7** *Let  $X_1 \geq \dots \geq X_n \geq 0$ ,  $\xi_1, \dots, \xi_n \geq 0$ . Then the following inequality for symmetric functions in  $\xi_1, \dots, \xi_n$*

$$\Psi_{123\dots n}^{112\dots n-1} \Psi_{1234\dots n}^{1223\dots n-1} \dots \Psi_{12\dots n-2}^{12\dots n-2 \ n-1 \ n-1} \geq \Psi_1^{1 \ 2\dots n-1} \Psi_1^{1 \ 2\dots n-1} \Psi_1^{1 \ 2\dots n-2 \ n} \dots \Psi_2^{1 \ 2\dots n-1} \Psi_2^{1 \ 2\dots n-1}$$

i.e.

$$\prod_{k=1}^{n-1} \Psi_1^{1 \ 2\dots k \ k \dots n} \geq \prod_{k=1}^n \Psi_1^{1 \ 2 \dots n-1} \Psi_1^{1 \ 2 \dots \hat{k} \dots n}$$

holds true.

We have checked this Conjecture 3.7 up to  $n = 5$  by using MAPLE and symmetric function package of J. Stembridge. For  $n$  bigger than five the computations are extremely intensive and hopefully in the near future would be possible by using more powerful computers.

Note that the right hand side of the Conjecture 3.7 involves symmetric functions of partial alphabets  $\xi_1, \xi_2, \dots, \xi_{k-1}, \xi_{k+1}, \dots, \xi_n$ . But the left hand side doesn't have this "defect". Our objective now is to give explicit formula for the right hand side in terms of the elementary symmetric functions of the full alphabet  $\xi_1, \xi_2, \dots, \xi_n$ . This we are going to achieve by using resultants as follows.

**Lemma 3.8** For any  $k$ , ( $1 \leq k \leq n$ ), we have

$$\Psi_{1 \dots \hat{k} \dots n}^{1 \dots k \dots n} = \sum_{j=0}^{n-1} a_j \xi_k^{n-1-j}$$

where

$$\begin{aligned} a_{n-1} &= 1 + X_1 e_1 + X_1 X_2 e_2 + \dots + X_1 \dots X_{n-1} e_{n-1}, \\ a_{n-2} &= -X_1 - X_1 X_2 e_1 - \dots - X_1 \dots X_{n-1} e_{n-2}, \\ &\dots \\ a_0 &= (-1)^{n-1} X_1 \dots X_{n-1} \end{aligned}$$

i.e.

$$a_{n-1-j} = (-1)^j \sum_{i=j}^{n-1} X_1 \dots X_i e_{i-j}$$

**Proof of Lemma 3.8.**

By definition we have

$$\Psi_{1 \dots \hat{k} \dots n}^{1 \dots n-1} = \sum_{i=0}^{n-1} X_1 \dots X_i e_i^{(k)} \quad (3.18)$$

where  $e_i^{(k)}$  is the  $i$ -th elementary function of  $\xi_1, \dots, \xi_{k-1}, \xi_{k+1}, \dots, \xi_n$ . Now from the decomposition

$$(1 + \xi_k t)^{-1} \prod_{j=1}^n (1 + \xi_j t) = \prod_{j \neq k} (1 + \xi_j t) = \sum_{i=0}^{n-1} e_i^{(k)} t^i$$

we get

$$e_i^{(k)} = e_i - e_{i-1} \xi_k + e_{i-2} \xi_k^2 - \dots + (-1)^i \xi_k^i$$

By substituting this into equation (3.18) the Lemma 3.8 follows. ■

Then, by Lemma 3.8, the right hand side

$$R_n = \prod_{k=1}^n \Psi_{1 \dots \hat{k} \dots n}^{1 \dots k \dots n-1} = \prod_{k=1}^n \left( \sum_{j=0}^{n-1} a_j \xi_k^{n-1-j} \right)$$

can be understood as a resultant  $R_n = \text{Res}(f, g)$  of the following two polynomials

$$\begin{aligned} f(x) &= \sum_{j=0}^{n-1} a_j x^{n-1-j} \\ g(x) &= \prod_{i=1}^n (x - \xi_i) = \sum_{j=0}^n (-1)^j e_j x^{n-j} \end{aligned}$$

The Sylvester formula

$$R_n = \begin{vmatrix} 1 & -e_1 & e_2 & -e_3 & \dots & (-1)^n e_n \\ & 1 & -e_1 & e_2 & -e_3 & \dots \\ & & \ddots & & & \\ & & & 1 & -e_1 & \dots \\ a_0 & a_1 & a_2 & \dots & a_n & \\ & a_0 & a_1 & a_2 & \dots & a_n \\ & & \ddots & & & \\ & & & a_0 & a_1 & a_2 & \dots & a_n \end{vmatrix} \quad \left( =: \begin{vmatrix} A & B \\ C & D \end{vmatrix} \right)$$

can be simplified as

$$= |A| \cdot |D - CA^{-1}B| = |D - CA^{-1}B|.$$

The entries of the  $n \times n$  matrix  $\Delta := D - CA^{-1}B$  are given by

$$\delta_{ij} = \begin{cases} \sum_{k=0}^{i-j} (-1)^{i+j} X_1 \cdots X_{i-k} e_{j-k} & \text{for } n-1 \geq i \geq j \geq 0 \\ \sum_{k=0}^n (-1)^{j-i} X_1 \cdots X_{j-i+k} e_{j-i+k+1} & \text{for } n-1 \geq j > i \geq 0 \end{cases}$$

For example, for  $n = 3$

$$\Delta_3 = \begin{vmatrix} 1 & X_1 e_2 + X_1 X_2 e_3 & -X_1 e_3 \\ -X_1 & 1 + X_1 e_1 & X_1 X_2 e_3 \\ X_1 X_2 & -X_1 - X_1 X_2 e_1 & 1 + X_1 e_1 + X_1 X_2 e_2 \end{vmatrix}$$

By elementary operations we get

$$\Delta_3 = \begin{vmatrix} 1 & * & * \\ 0 & \Psi_{123}^{112} & X_1(X_2 - X_1)e_3 \\ 0 & X_2 - X_1 & \Psi_{123}^{122} \end{vmatrix} = \begin{vmatrix} \Psi_{123}^{112} & X_1(X_2 - X_1)e_3 \\ X_2 - X_1 & \Psi_{123}^{122} \end{vmatrix}$$

Similarly, for  $n = 4$  we obtain

$$\Delta_4 = \begin{vmatrix} \Psi_{1234}^{1123} & -X_1(X_1 - X_2)e_3 - X_1 X_2(X_1 - X_3)e_4 & X_1(X_1 - X_2)e_4 \\ -(X_1 - X_2) & \Psi_{1234}^{1223} & -X_1 X_2(X_2 - X_3)e_4 \\ X_1(X_2 - X_3) & -(X_1 - X_3) - X_1(X_2 - X_3)e_1 & \Psi_{1234}^{1233} \end{vmatrix}$$

In general

$$\Delta_n = \det(\delta'_{ij})_{1 \leq i, j \leq n-1}$$

where

$$\delta'_{ij} = \begin{cases} \sum_{k=j+1}^{n-1} (-1)^{i+j+1} X_1 \cdots X_{k-1} (X_k - X_i) e_{k+1} , & \text{for } i < j \\ \Psi_{1 \ 2 \ \dots \ n}^1 \dots \overset{i}{2} \ \overset{i}{\dots} \ \overset{i}{n-1} , & \text{for } i = j \\ \sum_{k=0}^{j-1} (-1)^{i+j} X_1 \cdots X_{i-k-2} (X_{i-k-1} - X_i) e_k , & \text{for } i > j \end{cases}$$

**Corollary 3.9** *The conjecture 3.7 is equivalent to a Hadamard type inequality for the (non Hermitian) matrix  $(\delta'_{ij})_{1 \leq i, j \leq n-1}$ , i.e.*

$$\det(\delta'_{ij}) \leq \prod_{i=1}^{n-1} \delta'_{ii}.$$

## References

- [1] M. Atiyah. The geometry of classical particles. *Surveys in Differential Geometry* (International Press) **7** (2001).
- [2] M. Atiyah. Configurations of points. *Phil. Trans. R. Soc. Lond. A* **359** (2001), 1375–1387.
- [3] M. Atiyah and P. Sutcliffe. Polyhedra in Physics, Chemistry and Geometry, To appear in the Milan Journal of Mathematics
- [4] M. Atiyah and P. Sutcliffe. The geometry of point particles. arXiv:hep-th/0105179 v2, 15 Oct 2001.
- [5] M. Eastwood and P. Norbury. A proof of Atiyah’s conjecture on configurations of four points in Euclidean three-space. *Geometry & Topology* **5** (2001), 885–893.
- [6] D.Ž. Đoković, Proof of Atiyah’s conjecture for two special types of configurations, arXiv:math.GT/0205221 v4, 11 June 2002. *Electron. J. Linear Algebra* **9** (2002), 132–137.
- [7] D.Ž. Đoković, Verification of Atiyah’s conjecture for some nonplanar configurations with dihedral symmetry, arXiv:math.GT/0208089 v2, 13 Aug 2002.
- [8] I. G. Macdonald *Symmetric functions and Hall polynomials* 2<sup>nd</sup> edition, Oxford University Press, 1995.
- [9] A. F. Möbius Über eine Methode, um von Relationen, welche der Longimetrie angehören, zu entsprechenden Sätzen der Planimetrie zu gelangen, Ber. Verhandl. königl. Sächs. Ges. Wiss. Leipzig Math. Phys. Cl., 1852., 41–45.